Journal of Pure and Applied Algebra 44 (1987) 215–219
North-Holland

ON THE INTEGRAL COHOMOLOGY OF EXTRASPECIAL 2-GROUPS

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Communicated by E.M. Friedlander and S. Priddy
Received 6 May 1985

1. Introduction

Let V be a finite-dimensional vector space over F_2, the prime field of characteristic 2, V^* its dual and $S^*(V^*)$ the symmetric algebra over V^*. A quadratic form $Q(x) \in S^2(V^*)$ defines a central extension.

$$0 \to Z/2 \to V^{\sim} \xrightarrow{p} V \to 0.$$

The group $V^{\sim}$ is called an extraspecial 2-group. The mod 2 cohomology of $V^{\sim}$ was determined by Quillen [3] (see Section 2). Since $H^k(V^{\sim}; Z)$ is a finite abelian 2-group, $H^k(V^{\sim}; Z)$ is isomorphic to the direct sum of $A(k)$ and an elementary abelian 2-group, where $A(k)$ is the direct sum of $Z/2^{m(j)}$'s for $m(j) \geq 2$. The purpose of this paper is to determine the above $A(k)$.

Denote by $h(Q)$ the codimension of a maximal Q-isotropic subspace. For an integer k, denote the 2-exponent of k by $n(k)$, $k = 2^{n(k)} \cdot k'$ where k' is odd. Put

$$a(k,Q) = \begin{cases} n(k) & \text{if } n(k) \leq h(Q) - 1, \\ h(Q)+1 & \text{if } n(k) \geq h(Q). \end{cases}$$

If $h(Q) = 0$, $V^{\sim}$ is an elementary abelian 2-group, and if $h(Q) = 1$, $V^{\sim} = G \oplus A$, where G is the cyclic group of order 4 or the Dihedral group of order 4 and A is an elementary abelian 2-group. Since $H^*(V^{\sim}; Z)$ is well known in these cases, we therefore assume in this paper that $h(Q) \geq 2$. The main theorem of this paper is

Theorem 1.1. *If $h(Q) \geq 2$, then the abelian group $A(k)$ is a cyclic group of order $2^{a(k,Q)}$ if $k \equiv 0 \bmod 4$ and is equal to zero if $k \not\equiv 0 \bmod 4$.*

The paper is organized as follows: in Section 2 we review the result of Quillen [3] and the Sq^1-cohomology of $H^*(V^{\sim}; F_2)$, which is the E_2-term of the Bockstein spectral sequence, is determined by making use of a similar method to that of [2]. In Section 3 we determine the Bockstein spectral sequence and prove Theorem 1.1.

Remark. So far we haven't mentioned generators. One may hope Chern classes of some representation supply generators, but they don't in general. If a representation is trivial on $\ker p = Z/2$, every Chern class has at most order 2. If not, the only non-zero Stiefel–Whitney classes are of degree $2^h, 2^h - 2^i$ [3, 5.11]. Therefore almost all Chern classes of degree 2^j are $0 \bmod 2$.

Throughout this paper $H^*()$ denotes the mod 2 cohomology.

2. The Sq^1-cohomology of $H^*(V^{\sim})$

We write $\mathrm{Sq}^I = \mathrm{Sq}^{i(1)} \cdots \mathrm{Sq}^{i(n)}$ for any sequence of non-negative integers $I = (i(1), \ldots, i(n))$. If $I = (2^{n-1}, \ldots, 2^0)$, then Sq^I is simply denoted by $S(n)$.

We define elements $s(k)$ in $S^*(V^*)$. Let $s(1) = Q(x)$, $s(2) = B(x, x^2) = S(1)s(1)$, $\ldots, s(k) = B(x, x^{2^k}) = S(k-1)s(1)$, and $J(k)$ the ideal generated by $s(1), \ldots, s(k)$ in $S^*(V^*)$. The following is due to Quillen (see [3]):

Theorem 2.1. *If $h = h(Q)$, then the sequence $s(1), \ldots, s(h)$ is a regular sequence, $\operatorname{Ker} p^* = J(h)$ and $H^*(V^{\sim})$ is isomorphic to $S^*(V^*)/J(h) \otimes F_2[e]$, where e is the Euler class of some 2^h-dimensional real representation of $V^{\sim}$.*

In this section we will determine the Sq^1-cohomology of $H^*(V^{\sim})$. First we prove the following:

Lemma 2.2. $\mathrm{Sq}^1 e = 0$.

Proof. Since e is the Euler class of some real representation $\Delta : V \to \mathrm{O}(2^h)$,

$$\mathrm{Sq}^1 e = \mathrm{Sq}^1 w_{2^h}(\Delta) = w_{2^h}(\Delta) w_1(\Delta)$$

by Wu's formula. On the other hand $w_1(\Delta) = 0$ by 5.11 of Quillen [2] (note that $h \geq 2$). Therefore $\mathrm{Sq}^1 e = 0$.

Since $\operatorname{Im} p^* = S^*(V^*)/J(h)$ is a Sq^1-subcomplex of $H^*(V^{\sim})$ and $\mathrm{Sq}^1 e = 0$ by Lemma 2.1, as an algebra

$$H^*(H^*(V^{\sim}); \mathrm{Sq}^1) \cong H^*(S^*(V^*)/J(h); \mathrm{Sq}^1) \otimes F_2[u],$$

where u is represented by e.

Put $R(k) = S^*(V^*)/J(k)$. Since $\mathrm{Sq}^1 s(1) = s(2)$, $\mathrm{Sq}^1 s(2) = \mathrm{Sq}^1 \mathrm{Sq}^1 s(1) = 0$ and

$$\mathrm{Sq}^1 s(k+1) = \mathrm{Sq}^1 \mathrm{Sq}^{2^{k-1}} s(k) = \mathrm{Sq}^{2^{k-1}+1} s(k) = s(k)^2$$

for $k \geq 2$ by the Adem relation, we can show Sq^1 induces a derivation $d_k : R(k) \to R(k)$ for $k \geq 2$ (cf. [2]). First we determine $H^*(R(2))$.

Lemma 2.3. *$H^*(R(2)) = \Lambda(a)$ where* $\deg a = 3$.

Proof. Because $\mathrm{Sq}^1 s(2) = \mathrm{Sq}^1\mathrm{Sq}^1 s(1) = 0$, there exists an exact sequence of cochain complexes

$$0 \to \Sigma^3 S \xrightarrow{\cdot s(2)} S \to S/(s(2)) \to 0.$$

where $S = S^*(V^*)$ and $(\Sigma^3 S)_i = S_{i-3}$. This induces a long exact sequence

$$\cdots \to H^{j-3}(S) \to H^j(S) \to H^j(S/(s(2))) \xrightarrow{\delta} H^{j-2}(S) \to \cdots$$

Because $H^j(S) = 0$ for $j \neq 0$ and $H^0(S) = Z/2$, we get

$$H^j(S/(s(2))) = \begin{cases} 0, & j \neq 0, 2, \\ Z/2, & j = 0, 2. \end{cases}$$

Note that $s(1) \in S/(s(2))$ is a cocycle and $\delta[s(1)] = 1$, since $\mathrm{Sq}^1 s(1) = s(2)$. Consider another exact sequence of cochain complexes

$$0 \to \Sigma^2 S/(s(2)) \xrightarrow{\cdot s(1)} S/(s(2)) \to R(2) \to 0.$$

In the associated long exact sequence

$$\cdots \to H^{j-2}(S/(s(2))) \to H^j(S/(s(2))) \to H^j(R(2)) \to \cdots$$

$H^0(S/(s(2)) \xrightarrow{\delta} H^2(S/(s(2)))$ is an isomorphism since $H^2(S/s(2))$ is generated by $[s(1)]$. Now Lemma 2.3 is easily obtained.

Since $\mathrm{Sq}^1 s(k+1) = s(k)^2$ for each $k \geq 2$, we have exact sequences of cochain complexes

$$0 \to \Sigma^{2^k+1} R(k) \xrightarrow{\cdot s(k+1)} R(k) \to R(k+1) \to 0$$

for $2 \leq k \leq h-1$. They induce long exact sequences

$$\cdots \to H^{j-2^k-1}(R(k)) \to H^j(R(k)) \to H^j(R(k+1)) \xrightarrow{\delta} H^{j-2^k}(R(k)) \to \cdots$$

for $2 \leq k \leq h-1$ (cf. [2]). Now the following is easily obtained by induction on k.

Lemma 2.4. *For any k satisfying $2 \leq k \leq h-1$,*

(1) *as an algebra*

$$H^*(R(k)) = \Lambda(a, b(2), \ldots, b(k-1))$$

where $\deg a = 3$, $\deg b(j) = 2^j$,

(2) *there exists an element $b(k) \in H^{2^k}(R(k+1))$ such that $\delta(b(k)) = 1$ and $H^*(R(k+1))$ is a free $H^*(R(k))$ module generated by* 1 *and $b(k)$, and*

(3) $b(k)^2 = 0$.

Therefore we have

Theorem 2.5. *As an algebra*

$$H^*(H^*(V^\sim);\mathrm{Sq}^1)\cong\Lambda(a(Q),b(2,Q),\dots,b(h-1,Q))\otimes F_2[u(Q)],$$

where $\deg a(Q)=3$, $\deg b(j,Q)=2^j$, $\deg u(Q)=2^{h(Q)}$ ($a(Q)$, $b(j,Q)$ and $u(Q)$ are simply denoted by $a, b(j)$ and u when this will cause no confusion).

3. Proof of the main theorem

Consider the Bockstein spectral sequence $\{E_r\}$ associated with an exact couple

$$\begin{tikzcd} H^*(V^\sim;Z) \arrow[rr] & & H^*(V^\sim;Z) \arrow[dl] \\ & H^*(V^\sim) \arrow[ul] & \end{tikzcd}$$

It is well known that $\{E_r\}$ is a spectral sequence of algebra, where E_1 is isomorphic to $H^*(V^\sim)$ and E_2 is isomorphic to $H^*(H^*(V^\sim);\mathrm{Sq}^1)$ (cf. [1]). Note first that $H^*(V^\sim;Z)$ is a finite 2-group, so $E_r=\{1\}$ for a sufficiently large r. Also $H^{4k+1}(H^*(V^\sim);\mathrm{Sq}^1)=0$, so $b(2),\dots,b(h)$ and u are permanent cycles. By these facts we can easily prove the following.

Proposition 3.1. *For each Q, there exists a sequence of natural numbers*

$$1=r(1,Q)<\cdots<r(h,Q)$$

(h and $r(j,Q)$ are simply denoted by h and $r(j)$ when it causes no confusion) *such that*

(1) $E_{r(j)+1}=\cdots=E_{r(j+1)}=\Lambda(u(j),b(j+1),\dots,b(h-1))\otimes F_2[u]$ *and* $d_{r(j+1)}(u(j))=b(j+1)$ *for* $j\le h-2$, *where* $u(j)$ *is represented by* $ab(1)\cdots b(j)$,

(2) $E_{r(h-1)+1}=\cdots=E_{r(h)}=\Lambda(u(h-1))\otimes F_2[u]$ *and* $d_{r(h)}(u(h-1))=u$, *where* $u(h-1)$ *is represented by* $ab(1)\cdots b(h-1)$ *and*

(3) $E_{r(h)+1}=\cdots=E_\infty=\{1\}$.

On the other hand let W be a maximal Q-isotropic subspace of Q and $W^\sim=p^{-1}(W)$. Clearly W is an elementary abelian 2-group and the index of W in V is $2^{h(Q)}$. Consider the transfer

$$t: H^*(W^\sim;Z)\to H^*(V^\sim;Z)$$

and the restriction homomorphism

$$r: H^*(V^\sim;Z)\to H(W^\sim;Z).$$

As is well known $t\circ r=2^{h(Q)}$. Since $2x=0$ for any $x\in H^*(W^\sim;Z)$,

$$2^{h(Q)+1}y=2t\circ r(y)=t(2r(y))=0$$

for any $y\in H^*(V^\sim;Z)$. Therefore we have

Lemma 3.2. $r(h(Q)) \leq h(Q)+1$.

Let $(Q_0, (F_2)^2)$ be a hyperbolic plane, that is, $Q_0(x) = x_1 x_2$ for $x = (x_1, x_2)$. A new quadratic form $Q' = Q \oplus Q_0$ is defined over $V' = V \oplus (F_2)^2$. Put $h' = h(Q')$, $a' = a(Q')$, $b'(j) = b(j, Q')$, $u' = u(Q')$ and $r'(j) = r(j, Q')$. Now we compare these two. First $h' = h+1$. Consider the natural map

$$f^* : H^*(H^*((V')^{\sim}); \mathrm{Sq}^1) \to H^*(H^*(V^{\sim}); \mathrm{Sq}^1)$$

induced by the natural inclusion $f: V^{\sim} \to (V')^{\sim}$. By an easy computation we have $f^*(a') = a'$, $f^*(b'(j)) = b(j)$ for $1 \leq j \leq h-1$, $f^*(b'(h)) = 0$ and $f^*(u') = u^2$. Therefore $r(j) = r'(j)$ for $j \leq h-1$ and $r(h) > r'(h)$. Thus we have $r(h) \geq h+1$ and $r(h) = h+1$ if and only if $r(j+1) = r(j)+1$ for $j \leq h-2$ and $r(h) = r(h-1)+2$. Using Lemma 3.2, we have

Lemma 3.3.

$$r(j, Q) = \begin{cases} j & \textit{for } 1 \leq j \leq h-1, \\ h+1 & \textit{for } j = h. \end{cases}$$

Now Theorem 1.1 is an easy consequence of Proposition 3.1 and Lemma 3.3.

References

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